

# THE $G$ -HILBERT SCHEME FOR $\frac{1}{r}(1, a, r - a)$

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**ABSTRACT.** Following Craw, Maclagan, Thomas and Nakamura's work [Nak01],[CMT07b] on Hilbert schemes for abelian groups, we give an explicit description of the  $\text{Hilb}^G \mathbb{C}^3$  scheme for  $G = \langle \text{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a}) \rangle$  by a classification of all  $G$ -sets. We describe how the combinatorial properties of the fan of  $\text{Hilb}^G \mathbb{C}^3$  relates to the Euclidean algorithm.

## 1. INTRODUCTION

For any finite, abelian subgroup  $G$  of  $\text{GL}(n, \mathbb{C})$  of order  $r$ , Nakamura defines the  $G$ -Hilbert scheme  $\text{Hilb}^G \mathbb{C}^n$  as the irreducible component of the  $G$ -fixed set of the scheme  $\text{Hilb}^r \mathbb{C}^n$  which contains free orbits.

For such groups, the normalization of  $\text{Hilb}^G \mathbb{C}^n$  is a toric variety. The scheme  $\text{Hilb}^G \mathbb{C}^n$  is described in [Nak01] in terms of  $G$ -sets. In fact, the description is carried by a classification of  $G$ -sets.

There are several known cases when  $\text{Hilb}^G \mathbb{C}^n$  itself is a toric variety (i.e. it is normal): for  $n = 2$  and  $G \subset \text{GL}(2, \mathbb{C})$  by Kidoh [Kid01], for  $n = 3$  and  $G \subset \text{SL}(3, \mathbb{C})$  by Craw and Reid [CR02], for any  $n \geq 2$  and  $G = \langle \text{diag}(\varepsilon, \varepsilon^2, \varepsilon^4, \dots, \varepsilon^{2^n}) \rangle$  by Sebestean [Seb07]. In all these cases, if  $n \geq 3$  the quotient  $\mathbb{C}^n/G$  has canonical, non-terminal singularities.

Craw, Maclagan and Thomas in [CMT07b] describe  $\text{Hilb}^G \mathbb{C}^n$  for any finite, abelian group  $G \subset \text{GL}(n, \mathbb{C})$  in terms of initial ideals of some fixed monomial ideal by varying weight order. This gives a numerical method for finding the fan of  $\text{Hilb}^G \mathbb{C}^n$ .

In this paper, we use [Nak01] and [CMT07b] to give a conceptual description of  $\text{Hilb}^G \mathbb{C}^3$  scheme for any cyclic subgroup  $G \subset \text{GL}(3, \mathbb{C})$  for which the quotient  $\mathbb{C}^3/G$  is a terminal singularity; this is Theorem (6.2) below. By Morrison and Stevens [MS84], any such group is conjugated to a group generated by a diagonal matrix  $\text{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})$ , where  $a$  and  $r$  are any coprime natural numbers and  $\varepsilon$  is an  $r$ -th primitive root of unity.

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The description is carried out by classification of all possible  $G$ -sets in families, called triangles of transformations. These families correspond to steps in the Euclidean algorithm for  $b$  and  $r - b$ , where  $b$  is an inverse of  $a$  modulo  $r$  (see Main Theorem (6.2)). We prove that there are  $\frac{1}{2}(3r + b(r - b) - 1)$  different  $G$ -sets (see Theorem (6.4)).

We show that for  $a, r - a > 1$  the  $\text{Hilb}^G \mathbb{C}^3$  scheme is a normal variety with quadratic singularities. Note that  $\text{Hilb}^G \mathbb{C}^3$  for  $a = 1$  or  $r - a = 1$  is isomorphic to the Danilov resolution of  $\mathbb{C}^3/G$  singularity by [Kęd04].

The paper is organized as follows. Section 2 recalls basic definitions from [Nak01]. Section 3 contains classification of the  $G$ -sets by the number of valleys. It is used to show that the  $\text{Hilb}^G$  is normal. Section 4 contains definition of a primitive  $G$ -set. Every such  $G$ -set gives rise to a family of  $G$ -sets. The union of toric cones corresponding to  $G$ -sets in such family is called a triangle of transformations. In Section 5 we show how to obtain a new primitive  $G$ -set from another one. In Sections 6 the combinatoric properties of primitive  $G$ -sets and the triangles of transformations are related to the Euclidean algorithm. We show that all subcones of cones in all triangles of transformations form the fan of  $\text{Hilb}^G$  scheme. The formula counting the number of  $G$ -sets is given at the end of Section 6. Section 7 contains a concrete example of  $\text{Hilb}^G$  scheme for  $G \cong \mathbb{Z}_{14}$ .

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## 2. BASIC DEFINITIONS

Let us fix two coprime integers  $r, a \geq 2$ . Without loss of generality we may assume that  $a < r - a < r$ . Denote by  $G$  the cyclic group  $\mathbb{Z}_r$ , considered as a subgroup of  $\text{GL}(3, \mathbb{C})$ , generated by matrix  $\text{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})$ , where  $\varepsilon = e^{\frac{2\pi i}{r}}$ . The group  $G$  has  $r$  characters which may be identified with  $1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{r-1}$ .

We follow the notation of [Nak01]. Let  $N_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$  denote a free  $\mathbb{Z}$ -module with  $\mathbb{Z}$ -basis  $e_i$ . The lattice dual to  $N_0$  will be denoted  $M_0 = \text{Hom}_{\mathbb{Z}}(N_0, \mathbb{Z}) = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \oplus \mathbb{Z}e_3^*$ , where  $e_i^*(e_j) = \delta_{ij}$ . For the rest of this paper the variables  $x, y, z$  will be identified with  $e_1^*, e_2^*, e_3^*$  and a multiplicative notation will be used in the lattice  $M_0$ . For example, vector  $2e_1^* - e_3^*$  will be identified with the Laurent monomial  $x^2z^{-1}$ .

Let  $M_0^0$  be the positive octant in  $M_0$ , identified with monomials in the ring  $\mathbb{C}[x, y, z]$ . Set  $N = N_0 + \mathbb{Z}\frac{1}{r}(e_1 + ae_2 + (r - a)e_3)$  and let  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  be a dual lattice. Lattice  $M$  will be identified with a sublattice of  $M_0$  consisting of  $G$ -invariant Laurent monomials. When no confusion arise, vector  $a_1e_1 + a_2e_2 + a_3e_3$  will be denoted  $(a_1, a_2, a_3)$ . For example  $\frac{1}{5}(1, 2, 3)$  stands for  $\frac{1}{5}e_1 + \frac{2}{5}e_2 + \frac{3}{5}e_3$ .

Let  $G^\vee$  denote the character group of  $G$ . The group  $G$  acts on the left on regular functions on  $\mathbb{C}^3$  by setting  $(g \cdot f)(p) = f(g^{-1}p)$ , where

$g \in G$ ,  $p \in \mathbb{C}^3$  and  $f$  is a regular function on  $\mathbb{C}^3$ . This action can be extended to the lattice  $M_0$  (by identifying  $M_0$  with the lattice of exponents of Laurent monomials in  $x, y, z$ ). Thus, we have the natural grading:

$$M_0 = \bigoplus_{\chi \in G^\vee} M_0^\chi.$$

**Definition 2.1.** Let  $\text{wt} : M_0 \longrightarrow G^\vee$  denote group homomorphism sending an element of the lattice  $M_0$  to its grade.

We will denote by  $m \bmod n$  an integer  $k \in 0, \dots, n-1$  such that  $n \mid (m - k)$ .

**Definition 2.2.** (Nakamura) A subset  $\Gamma$  of monomials in  $\mathbb{C}[x, y, z]$  is called a  $G$ -set if

- (1) it contains the constant monomial 1,
- (2) if  $vw \in \Gamma$  then  $v \in \Gamma$  and  $w \in \Gamma$ ,
- (3) the restriction of the function  $\text{wt}$  to  $\Gamma$  is a bijection.

**Remark.** Since  $\text{wt}(1) = \text{wt}(yz)$ , it follows that  $yz \notin \Gamma$  for any  $G$ -set  $\Gamma$ . Hence the monomials in  $\Gamma$  are of the form  $x^*y^*$  and  $x^*z^*$ , where  $*$  stands for any nonnegative integer.

**Definition 2.3.** For any  $G$ -set  $\Gamma$  define  $i(\Gamma), j(\Gamma), k(\Gamma)$  to be the unique nonnegative integers such that

$$\begin{aligned} x^{i(\Gamma)} &\in \Gamma, x^{i(\Gamma)+1} \notin \Gamma, \\ y^{j(\Gamma)} &\in \Gamma, y^{j(\Gamma)+1} \notin \Gamma, \\ z^{k(\Gamma)} &\in \Gamma, z^{k(\Gamma)+1} \notin \Gamma. \end{aligned}$$

When no confusion arise we write for short:

$$\begin{aligned} i &= i(\Gamma), \\ j &= j(\Gamma), \\ k &= k(\Gamma). \end{aligned}$$

**Definition 2.4.** (Nakamura) A monomial  $x^m y^n$  (resp.  $x^m z^n$ ) for  $m, n \geq 0$  is called a  $y$ -valley (resp.  $z$ -valley) for  $\Gamma$ , if

$$\begin{aligned} x^m y^n, x^{m+1} y^n, x^m y^{n+1} &\in \Gamma \quad \text{but} \quad x^{m+1} y^{n+1} \notin \Gamma \\ (\text{resp. } x^m z^n, x^{m+1} z^n, x^m z^{n+1} &\in \Gamma \quad \text{but} \quad x^{m+1} z^{n+1} \notin \Gamma). \end{aligned}$$

We call a  $y$ -valley or  $z$ -valley a valley for brevity.

**Definition 2.5.** For any  $v \in M_0^0$  let  $\text{wt}_\Gamma(v)$  denote the unique  $w \in \Gamma$  such that  $\text{wt}(v) = \text{wt}(w)$ .

3. CLASSIFICATION OF  $G$ -SETS

In this section we show that any  $G$ -set has at most one  $y$ -valley and at most one  $z$ -valley. Following Nakamura, for every  $G$ -set we construct a semigroup  $S(\Gamma)$  in the lattice  $M$  and prove that it is saturated. It turns out that the  $G$ -sets correspond to the cones of maximal dimension in the fan of  $\text{Hilb}^G \mathbb{C}^3$ .

**Remark 3.1.** The following statements are immediate from the definitions:

- (1) if  $\text{wt}_\Gamma(v) = w$ ,  $v \notin \Gamma$  and  $u \cdot w \in \Gamma$ , then  $u \cdot v \notin \Gamma$ ,
- (2) if  $\text{wt}_\Gamma(v) = w$ , then  $\text{wt}_\Gamma(u \cdot v) = u \cdot w$  for any  $u \in M_0$  such that  $u \cdot w \in \Gamma$ ,
- (3) if  $\text{wt}_\Gamma(v) = w$ ,  $u \in M$  then  $\text{wt}_\Gamma(u \cdot v) = w$ .

**Corollary 3.2.** Let  $\Gamma$  be a  $G$ -set and  $v \in M_0^0 - \Gamma$ . If  $x^{-1} \cdot v \in \Gamma$  (resp.  $y^{-1} \cdot v \in \Gamma$ ,  $z^{-1} \cdot v \in \Gamma$ ) then  $\text{wt}_\Gamma(v) = w$ , where  $w \in \Gamma$  but  $x^{-1} \cdot w \notin \Gamma$  (resp.  $z \cdot w \notin \Gamma$ ,  $y \cdot w \notin \Gamma$ ).

*Proof.* Use observation (1) and (3) from Remark (3.1).  $\square$

**Lemma 3.3.** A  $G$ -set can only have 0, 1 or 2 valleys.

*Proof.* Suppose that  $x^m y^n$  is a  $y$ -valley for  $\Gamma$ . Then  $v = x^{m+1} y^{n+1}$  satisfies assumptions of Corollary (3.2). Hence,  $x^{-1} \cdot \text{wt}_\Gamma(v) \notin \Gamma$  and  $z \cdot \text{wt}_\Gamma(v) \notin \Gamma$ , so  $\text{wt}_\Gamma(v) = z^{k(\Gamma)}$ . Therefore,  $G$ -set  $\Gamma$  has at most one  $y$ -valley, and, analogously at most one  $z$ -valley.  $\square$

**Corollary 3.4.** Suppose that  $G$ -set  $\Gamma$  has  $y$ -valley  $w$  and  $z$ -valley  $v$ . Then

$$\begin{aligned} \text{wt}_\Gamma(y^{j(\Gamma)+1}) &= x \cdot w, \\ \text{wt}_\Gamma(z^{k(\Gamma)+1}) &= x \cdot v. \end{aligned}$$

*Proof.* Use observation (2) from Remark (3.1).  $\square$

**Notation 3.5.** From now on we will usually denote by  $i_y, j_y$  the exponents of the  $y$ -valley  $x^{i_y} y^{j_y}$  and by  $i_z, k_z$  the exponents of the  $z$ -valley  $x^{i_z} z^{k_z}$  of some fixed  $G$ -set  $\Gamma$ .

**Lemma 3.6.** The only possible  $G$ -sets with no valleys are:

$$\begin{aligned} \Gamma^x &= \{1, x, \dots, x^{r-1}\}, \\ \Gamma_l^{yz} &= \{y^{r-l-1}, \dots, y, 1, z, \dots, z^l\} \text{ for } l = 0, \dots, r-1. \end{aligned}$$

*Proof.* Let  $i, j, k$  be integers like in Definition (2.3). Corollary (3.2) shows that  $\text{wt}_\Gamma(y^{j+1}) = x^{i'} z^k$ , for some  $i' \geq 0$ . If  $i' = 0$ , then  $\text{wt}(z^{k+1}) = \text{wt}(y^j)$  and since  $a, r$  are coprime, it follows that  $j = r - k - 1$ , hence  $i = 0$ . Consider the case  $i' > 0$ . Then  $\text{wt}_\Gamma(x^{i'-1} z^{k+1}) = x^{i''} y^j$  by Corollary (3.2). It follows immediately that  $i'' = i = r - 1$  and so  $j = k = 0$ .  $\square$

**Lemma 3.7.** *Let  $\Gamma$  be a  $G$ -set with exactly one valley. If  $\Gamma$  has  $y$ -valley equal to  $x^{i_z} z^{k_z}$ , then*

$$\text{wt}_\Gamma(x^{i+1}) = z^{k-k_z},$$

$$\text{wt}_\Gamma(z^{k+1}) = x^{i-i_z} y^j.$$

*If  $G$ -set  $\Gamma$   $z$ -valley equal to  $x^{i_y} y^{j_y}$ , then*

$$\text{wt}_\Gamma(x^{i+1}) = y^{j-j_y},$$

$$\text{wt}_\Gamma(y^{j+1}) = x^{i-i_y} z^k.$$

*Proof.* We prove the lemma in the case of  $z$ -valley  $w = x^{i_z} z^{k_z}$ . The monomial  $\text{wt}_\Gamma(z^{k+1})$  is of the form  $x^l y^j$ , where  $0 \leq l \leq i$ . Noting that  $\text{wt}_\Gamma(xz \cdot w) = y^j$  we get  $l = i - i_z$ . It follows that the monomials  $x^{i-i_z} y^j$  and  $z^{k+1}$  are of the same weight, therefore  $\text{wt}_\Gamma(z^{k+1}) = x^{i-i_z} y^j$ .  $\square$

**Lemma 3.8.** *Let  $\Gamma$  be a  $G$ -set with two valleys  $v, w$ , where*

$$v = x^{i_y} y^{j_y},$$

$$w = x^{i_z} z^{k_z}.$$

*Then  $i_y + i_z + 1 = i$ , and*

$$\text{wt}_\Gamma(x^{i+1}) = \begin{cases} y^{(j-j_y)-(k-k_z)} & \text{if } (j-j_y) - (k-k_z) \geq 0, \\ z^{(k-k_z)-(j-j_y)} & \text{otherwise.} \end{cases}$$

*Proof.* Let  $u$  be a monomial such that  $u \notin \Gamma$  and  $x^{-1}u \in \Gamma$ . Then  $\text{wt}_\Gamma(u) = z^l$  for some  $0 \leq l \leq k$  or  $\text{wt}_\Gamma(u) = y^l$  for  $0 \leq l \leq j$ . We know already that  $\text{wt}_\Gamma(xz \cdot w) = y^j$  and  $\text{wt}_\Gamma(xy \cdot v) = z^k$ , which implies that  $\text{wt}_\Gamma(x^{i+1}) = y^{(j-j_y)-(k-k_z)}$  if  $(j-j_y) - (k-k_z) \geq 0$  and  $\text{wt}_\Gamma(x^{i+1}) = z^{(k-k_z)-(j-j_y)}$  otherwise. The monomial  $x^{i_y+i_z+1}$  has the same weight as  $x^{i+1}$  hence they are equal.  $\square$

**Definition 3.9.** (Nakamura) *For any  $v \in M_0$  and a  $G$ -set  $\Gamma$  define (using a multiplicative notation in the lattice  $M_0$ )*

$$s_\Gamma(v) = v \text{wt}_\Gamma^{-1}(v).$$

*We will write it simply  $s(v)$  when no confusion can arise. Define the cones*

$$\sigma(\Gamma) = \{\alpha \in N_0 \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \alpha, s_\Gamma(v) \rangle \geq 0, \quad \forall v \in M_0^0\},$$

$$\sigma^\vee(\Gamma) = \{v \in M_0 \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \alpha, v \rangle \geq 0, \quad \forall \alpha \in \sigma(\Gamma)\},$$

*where  $\langle \cdot, \cdot \rangle$  denotes the pairing between  $N_0$  and  $M_0$ .*

*Let  $S(\Gamma)$  be a subsemigroup of the lattice  $M$ , generated by the set  $\{s_\Gamma(v) \in M \mid v \in M_0^0\}$  as a semigroup. Set*

$$V(\Gamma) = \text{Spec } \mathbb{C}[S(\Gamma)].$$

Note that

$$\mathbb{C}[S(\Gamma)] \subset \mathbb{C}[\sigma^\vee(\Gamma) \cap M].$$

Moreover, the cones  $\sigma(\Gamma), \sigma^\vee(\Gamma)$  are dual to each other and the cone  $\sigma^\vee(\Gamma) \cap M$  is the saturation of the semigroup  $S(\Gamma)$  in the lattice  $M$ . It will follow from Lemma (3.11) that  $S(\Gamma)$  is finitely generated as a semigroup.

**Theorem 3.10** (Nakamura). *Let  $G$  be a finite abelian subgroup of  $\mathrm{GL}(3, \mathbb{C})$ . When  $\Gamma$  varies through all  $G$ -sets the set of all faces of all 3-dimensional cones  $\sigma(\Gamma)$  forms a fan in lattice  $N \otimes \mathbb{R}$  supported on the positive octant. Toric variety defined by this fan is isomorphic to the normalization of the  $\mathrm{Hilb}^G \mathbb{C}^3$  scheme (see [Nak01, Theorem 2.11] and [CMT07a, §5]). Moreover, the affine varieties  $V(\Gamma)$  form an open covering of the  $\mathrm{Hilb}^G \mathbb{C}^3$  scheme when  $\Gamma$  varies through all  $G$ -sets.*

**Lemma 3.11.** (Nakamura) *Let  $A \subset M_0^0 - \Gamma$  be a finite set such that  $M_0^0 - \Gamma = A \cdot M_0^0$ . If  $\sigma(\Gamma)$  is a 3-dimensional cone then  $S(\Gamma)$  is generated by the finite set  $\{s_\Gamma(v) \mid v \in A\}$  as a semigroup (see [Nak01, Lemma 1.8]).*

**Remark 3.12.** Note that Theorem (3.10) and Lemma (3.11) are stated in [Nak01] without the assumption on dimension of  $\sigma(\Gamma)$  in which case they are false. A counterexample and a correction can be found in [CMT07b, Example 4.12 and Theorem 5.2].

**Lemma 3.13.** *Suppose that  $\Gamma$  is a  $G$ -set in the case of  $\frac{1}{r}(1, a, r - a)$  action. Then the cone  $\sigma(\Gamma)$  is 3-dimensional. Moreover, if  $\Gamma$  has 0 or 1 valley then  $S(\Gamma) \cong \mathbb{C}[x, y, z]$ . If  $\Gamma$  has 2 valleys then  $S(\Gamma) \cong \mathbb{C}[x, y, z, w]/(xy - zw)$ .*

*Proof.* The lemma will be proven only in the case of a  $G$ -set with 2 valleys as the method carries over to the other cases.

Suppose that  $\Gamma$  is a  $G$ -set with 2 valleys,  $v = x^{i_y} y^{j_y}$ ,  $w = x^{i_z} z^{k_z}$  and set

$$\begin{aligned} \alpha &= x^{i+1}, \\ \beta &= y^{j+1}, \\ \gamma &= z^{k+1}, \\ \delta_y &= xy \cdot v, \\ \delta_z &= xz \cdot w, \end{aligned}$$

where  $i, j, k$  are the largest exponents such that  $x^i, y^j, z^k$  belong to  $\Gamma$ . We will start by showing that  $s(\beta), s(\gamma), s(\delta_y)$  and  $s(\delta_z)$  generate semigroup  $S(\Gamma)$ . Assume that  $u \in M_0^0$ ,  $t = x, y$  or  $z$  and note that

$$s(t \cdot u) = s(u)s(t \cdot \mathrm{wt}_\Gamma(u)).$$

By the above formula it suffices to show that for any  $u \in \Gamma$  such that  $t \cdot u \notin \Gamma$  the Laurent monomial  $s(t \cdot u)$  can be expressed as a

product of  $s(\beta), s(\gamma), s(\delta_y)$  and  $s(\delta_z)$  with nonnegative exponents. By Lemma (3.8):

$$s(\alpha) = \begin{cases} x^{i+1}y^{-(j-j_y)+(k-k_z)} & \text{if } (j-j_y) \geq (k-k_z), \\ x^{i+1}z^{(j-j_y)-(k-k_z)} & \text{otherwise,} \end{cases}$$

$$s(\beta) = xy^{-(j+1)} \cdot w,$$

$$s(\gamma) = xz^{-(k+1)} \cdot v,$$

$$s(\delta_y) = xyz^{-k} \cdot v,$$

$$s(\delta_z) = xy^{-j}z \cdot w,$$

hence

$$s(\beta)s(\delta_z) = s(\gamma)s(\delta_y) = s(yz),$$

$$s(\alpha) = \begin{cases} s(\delta_y)s(\delta_z)(yz)^{j-j_y-1} & \text{if } (j-j_y) \geq (k-k_z), \\ s(\delta_y)s(\delta_z)(yz)^{k-k_z-1} & \text{otherwise.} \end{cases}$$

Let  $u \in \Gamma$  and  $y \cdot u \notin \Gamma$ . If  $u = x^l y^j$ , where  $l = 0, \dots, i_y$  then  $s(y \cdot u) = s(\beta)$ . If  $u = x^l y_y^j$ , where  $l = i_y + 1, \dots, i$  then  $s(y \cdot u) = s(\delta_y)$ . Analogously  $s(z \cdot u)$  is equal to  $s(\gamma)$  or to  $s(\delta_y)$  for any  $u \in \Gamma, z \cdot u \notin \Gamma$ .

It remains to consider  $u \in \Gamma$  such that  $x \cdot u \notin \Gamma$ . Observe that  $\text{wt}_\Gamma(x \cdot u)$  is of the form  $y^l$  or  $z^l$  for some positive  $l$  ( $l = 0$  can happen only if  $\Gamma = \Gamma^\times$ ). If  $u' = y^{-1}u \in \Gamma$  then  $x \cdot u' \notin \Gamma$  and

$$s(x \cdot u) = s(y \cdot xu') = s(xu')s(y \text{wt}_\Gamma(x \cdot u')) = s(xu')(yz)^n, \text{ where } n = 0, 1.$$

By induction for any such  $u \in \Gamma$  the monomial  $s(x \cdot u)$  is equal to  $p \cdot (xy)^m$ , where  $m > 0$  and  $p = s(\alpha), s(\delta_y)$  or  $s(\delta_z)$ .

This shows that  $S(\Gamma)$  is generated by  $s(\beta), s(\gamma), s(\delta_y)$  and  $s(\delta_z)$ . To conclude it is enough to show that some (in fact any) 3 out of 4 generators form a  $\mathbb{Z}$ -basis of the lattice  $M$ . This is implied by computing the following determinant, using equality from Lemma (3.8):

$$\begin{vmatrix} -i_z - 1 & j + 1 & -k_z \\ i_y + 1 & j_y + 1 & -k \\ -i_y - 1 & -j_y & k + 1 \end{vmatrix} = r.$$

□

**Corollary 3.14.** The semigroup  $S(\Gamma)$  coincides with the semigroup algebra  $\mathbb{C}[\sigma^\vee(\Gamma') \cap M]$  for any  $G$ -set. In particular,  $\text{Hilb}^G \mathbb{C}^3$  is normal.

#### 4. $G$ -IGSAW TRANSFORMATIONS

To get an effective description of the fan of the  $\text{Hilb}^G$  scheme, we introduce Nakamura's  $G$ -igsaw transformation, which will allow to organize  $G$ -sets in families and to explain how these are related to each other.

$G$ -igsaw transformation is a method of constructing a new  $G$ -set from the other. In fact, two  $G$ -sets  $\Gamma$  and  $\Gamma'$  are related by a  $G$ -igsaw

transformation if and only if the cones  $\sigma(\Gamma)$  and  $\sigma(\Gamma')$  share a 2-dimensional face.

When reading Sections 4 through 6, it may be useful for a reader to consult an example provided in Section 7.

**Lemma 4.1.** (Nakamura) *Let  $\Gamma$  be a  $G$ -set for the action of type  $\frac{1}{r}(1, a, r - a)$  and let  $\tau$  be a 2-dimensional face of  $\sigma(\Gamma)$ . There exist two monomials  $u \in M_0^0$  and  $v \in \Gamma$  such that*

- (1)  $v = \text{wt}_\Gamma(u)$ ,
- (2)  $u, v$  do not have common factors in  $M_0^0$ ,
- (3)  $uv^{-1}$  is a primitive monomial,
- (4)  $\tau = \sigma(\Gamma) \cap (uv^{-1})^\perp$ ,

*Proof.* This is a particular case of [Nak01, Lemma 2.5] □

**Definition 4.2.** (Nakamura) *Let  $\Gamma$  be a  $G$ -set and let  $\tau$  be a 2-dimensional face of  $\sigma(\Gamma)$ . Suppose that monomials  $u, v$  given by Lemma (4.1) are not equal to 1 and set  $c(w) = \max\{c \in \mathbb{Z} \mid wv^{-c} \in M_0^0\}$  for any  $w \in \Gamma$ . We define the  $G$ -igsaw transformation of  $\Gamma$  in the direction of  $\tau$  to be the set*

$$\Gamma' = \{w \cdot u^{c(w)} v^{-c(w)} \mid w \in \Gamma\}.$$

**Lemma 4.3.** (Nakamura) *The  $G$ -igsaw transformation of a  $G$ -set is a  $G$ -set.*

*Proof.* See [Nak01, Lemma 2.8] □

**Lemma 4.4.** *Suppose that  $\Gamma$  is a  $G$ -set for the action  $\frac{1}{r}(1, a, r - a)$ . Let  $\alpha = x^{i+1}, \beta = y^{j+1}, \gamma = z^{k+1}$ , where  $i, j, k$  are the maximal exponents such that  $x^i, y^j, z^k \in \Gamma$ . Let  $\tau$  be a 2-dimensional face of  $\sigma(\Gamma)$  and let  $u$  be the monomial given by Lemma (4.1). If  $\Gamma$  has 0 or 1 valley then  $u = \alpha, \beta$  or  $\gamma$ . If  $\Gamma$  has 2 valleys then  $u = \beta, \gamma, \delta_y$  or  $\delta_z$ , where  $\delta_y$  is equal to the  $y$ -valley of  $\Gamma$  multiplied by  $xy$  and  $\delta_z$  is equal to the  $z$ -valley of  $\Gamma$  multiplied by  $xz$ .*

*Proof.* Suppose that  $\Gamma$  has one valley and  $\tau$  is a face of  $\sigma(\Gamma)$  dual to the ray of  $\sigma^\vee(\Gamma)$  spanned by  $s(\alpha)$ . The 1-dimensional lattice  $M \cap \tau^\perp$  has 2 generators. Therefore  $uv^{-1}$  is equal either to  $s(\alpha)$  or  $s(\alpha)^{-1}$ . Clearly, the only choice is  $u = \alpha, v = \text{wt}_\Gamma(\alpha)$ . Suppose that  $d \in M_0^0$  is a common factor of  $u$  and  $v$ . Then both  $ud^{-1}, vd^{-1}$  belong to  $\Gamma$  and they are of the same weight. Hence  $d = 1$ . □

**Definition 4.5.** *Let  $\Gamma$  be a  $G$ -set with 0 or 1 valley and let  $\tau$  be the 2-dimensional face of  $\sigma(\Gamma)$ . The  $G$ -igsaw transformation of  $\Gamma$  in the direction of  $\tau$  is called upper (resp. right, left) transformation if  $u = \alpha$  (resp.  $u = \beta, u = \gamma$ ), where the monomial  $u$  is as in Lemma (4.1). The upper, left and right transformations of  $\Gamma$  will be denoted by  $T_U(\Gamma), T_R(\Gamma)$  and  $T_L(\Gamma)$ , respectively.*



By slight abuse of notation, the  $G$ -igsaw transformation of  $G$ -set  $\Gamma$  with 2 valleys is called left (resp. upper left, right, left) transformation if the corresponding monomial  $u$  is equal to  $\beta$  (resp.  $\gamma, \delta_y, \delta_z$ ). The right, left, upper right and upper left  $G$ -igsaw transformations of  $\Gamma$  will be denoted by  $T_{UR}(\Gamma), T_{UL}(\Gamma), T_R(\Gamma), T_L(\Gamma)$ , respectively.

**Definition 4.6.** We say that a  $G$ -set  $\Gamma$  is spanned by monomials  $u_1, \dots, u_n$  if  $\Gamma$  consists of all monomials dividing  $u_1, \dots, u_n$ . If  $G$ -set  $\Gamma$  is spanned by monomials  $u_1, \dots, u_n$  we write

$$\Gamma = \text{span}(u_1, \dots, u_n).$$

**Lemma 4.7.** Let  $\Gamma = \text{span}(x^{i_y}y^j, x^i z^k)$ , where  $i_y < i$  (resp. let  $\Gamma = \text{span}(x^i y^j, x^{i_k} z^k)$ , where  $i_z < i$ ) be a  $G$ -set with one  $y$ -valley equal to  $x^{i_y}$  (resp. one  $z$ -valley equal to  $x^{i_z}$ ).

Then

$$\begin{aligned} T_U(\Gamma) &= \text{span}(x^{i+i_y+1}, x^{i_y}y^{j-1}, x^i z^k) \\ (\text{resp. } T_U(\Gamma) &= \text{span}(x^{i+i_z+1}, x^i y^j, x^{i_k} z^{k-1})). \end{aligned}$$

In particular, the upper transformation of  $\Gamma$  has

- no valleys if and only if  $j = 1, k = 0$  (resp.  $j = 0, k = 1$ ). In fact, in this case  $T_U(\Gamma) = \Gamma^x$ .
- one  $z$ -valley (resp. one  $y$ -valley) if and only if  $j = 1, k > 0$  (resp.  $j > 0, k = 1$ ). In both cases the valley is equal to  $x^i$ .
- two valleys: the  $y$ -valley equal to  $x^{i_y}$  and the  $z$ -valley equal to  $x^i$  (resp. the  $y$ -valley equal to  $x^i$  and the  $z$ -valley equal to  $x^{i_z}$ ) in the remaining cases.

*Proof.* The upper transformation is obtained by replacing each monomial  $w \in \Gamma$ , divisible by  $y^j$  (resp. by  $z^k$ ) by the monomial  $x^{n(i+1)}y^{-nj} \cdot w$  for some  $n \geq 1$ . The proof is straightforward.  $\square$

**Lemma 4.8.** Let  $\Gamma$  be a  $G$ -set with 2 valleys:  $y$ -valley equal to  $v = x^{i_y}y^{j_y}$  and  $z$ -valley equal to  $w = x^{i_z}z^{k_z}$ . Assume that  $\Gamma$  is spanned by  $x^i y^{j_y}, x^i z^{k_z}, x^{i_y}y^j, x^{i_z}z^k$ . Let  $T$  stand for right, left, upper right or upper left transformation.

Then  $T(\Gamma)$  is spanned by:

$$\begin{array}{llll} x^i y^{j_y}, & x^i z^{k_z-1}, & x^{i_y} y^{j+1}, & x^{i_z} z^k & T = T_R, k_z \geq 1 \\ x^i y^{j_y-1}, & x^i z^{k_z}, & x^{i_y} y^j, & x^{i_z} z^{k+1} & T = T_L, j_y \geq 1 \\ x^i y^{j_y+1}, & x^i z^{k_z}, & x^{i_y} y^j, & x^{i_z} z^{k-1} & \text{if } T = T_{UR}, \\ x^i y^{j_y}, & x^i z^{k_z+1}, & x^{i_y} y^{j-1}, & x^{i_z} z^k & T = T_{UL}. \end{array}$$

*Proof.* The proof is a matter of straightforward computation. It follows directly by considering each case separately cf. Lemma(4.4).  $\square$

Note that the  $G$ -igsaw transformation of a  $G$ -set with two valleys may have only one valley.

**Corollary 4.9.** Let  $\Gamma$  be a  $G$ -set spanned by  $x^i y^{j_y}, x^i z^{k_z}, x^{i_y} y^j, x^{i_z} z^k$  with 2 valleys:  $y$ -valley equal to  $v = x^{i_y} y^{j_y}$  and  $z$ -valley equal to  $w = x^{i_z} z^{k_z}$ . If  $j_y, k_z \geq 1$  then

$$\begin{aligned} T_R(T_{UL}(\Gamma)) &= \Gamma, & T_{UL}(T_R(\Gamma)) &= \Gamma, \\ T_L(T_{UR}(\Gamma)) &= \Gamma, & T_{UR}(T_L(\Gamma)) &= \Gamma, \end{aligned}$$

that is right and upper left (resp. left and upper right) transformations are inverse operations. Moreover, if  $j, k, j - j_y, k - k_z \geq 2$  then

$$T_{UL}(T_{UR}(\Gamma)) = T_{UR}(T_{UL}(\Gamma)),$$

that is upper left and upper right transformations commute.

**Corollary 4.10.** Let  $\Gamma$  be a  $G$ -set spanned by  $x^i y^{j_y}, x^i z^{k_z}, x^{i_y} y^j, x^{i_z} z^k$ , with 2 valleys:  $y$ -valley equal to  $v = x^{i_y} y^{j_y}$  and  $z$ -valley equal to  $w = x^{i_z} z^{k_z}$ . Let  $\Gamma' = T_{UR}^m(T_{UL}^n(\Gamma))$ , where  $m + n \leq \min\{j, k, j - j_y, k - k_z\}$ . Then  $\Gamma'$  is spanned by  $x^i y^{j_y+m}, x^i z^{k_z+n}, x^{i_y} y^{j-n}, x^{i_z} z^{k-m}$ . If  $m + n < \min\{j, k, j - j_y, k - k_z\}$  then  $\Gamma'$  has two valleys. If  $m + n = \min\{j, k, j - j_y, k - k_z\}$  then  $\Gamma'$  has one valley (one of the monomials  $x^i y^{j_y+m}, x^i z^{k_z+n}, x^{i_y} y^{j-n}, x^{i_z} z^{k-m}$  spanning  $\Gamma'$  is redundant).

## 5. TRIANGLES OF TRANSFORMATIONS AND PRIMITIVE $G$ -SETS

In this section we introduce primitive  $G$ -sets, which have a particular shape. Every primitive  $G$ -set such gives rise to a family of  $G$ -sets, called here a triangle of transformations. It will turn out that most  $G$ -sets belong to some triangle of transformations. We define a sequence of primitive  $G$ -sets containing every primitive  $G$ -set for fixed integers  $r$  and  $a$ .

**Definition 5.1.** Let  $\Gamma$  be a  $G$ -set with two valleys, spanned by  $x^i y^{j_y}, x^i z^{k_z}, x^{i_y} y^j, x^{i_z} z^k$ . The set

$$\Theta(\Gamma) = \{T_{UR}^m(T_{UL}^n(\Gamma)) \mid m + n \leq \min\{j, k, j - j_y, k - k_z\}\}$$

will be called triangle of transformations of  $\Gamma$ .

The union of the supports of  $G$ -sets belonging to the set  $\Theta(\Gamma)$  is a simplicial cone (see Corollary (5.13)), hence we call  $\Theta(\Gamma)$  a triangle of transformations.

**Definition 5.2.** A  $G$ -set  $\Gamma$  is called primitive if it has a  $y$ -valley equal to  $x^{i_y}$  and a  $z$ -valley equal to  $x^{i_z}$  for some nonnegative  $i_y, i_z$ .

The name primitive is justified by the fact that every  $G$ -set with two valleys belong to a triangle of transformations of some primitive  $G$ -set. This fact will follow from the Main Theorem.

**Definition 5.3.** For fixed coprime integers  $r, a$  define let  $\Gamma_1$  be a  $G$ -set spanned by  $x, y^{b-1}, z^{r-b-1}$ , where  $b \in \{1, \dots, r-1\}$  is as an inverse of  $a$  modulo  $r$ .

The  $G$ -set  $\Gamma_1$  is primitive and the monomial  $x$  is simultaneously its  $y$ -valley and  $z$ -valley.

**Lemma 5.4.** *Let  $\Gamma$  be a primitive  $G$ -set spanned by  $x^i, x^{i_y}y^j, x^{i_z}z^k$ . Then  $\Theta(\Gamma)$  consists of  $\binom{\min\{j,k\}+2}{2}$   $G$ -sets.*

*Proof.* It is clear from definition of  $\Theta(\Gamma)$ .  $\square$

**Lemma 5.5.** *Let  $\Gamma$  be a primitive  $G$ -set spanned by  $x^i, x^{i_y}y^j, x^{i_z}z^k$ . Suppose that  $j < k$  (resp.  $k < j$ ). The  $G$ -set  $T_U(T_{UR}^j(\Gamma))$  (resp.  $T_U(T_{UL}^k(\Gamma))$ ) is spanned by  $x^{i+i_z+1}, x^i y^j, x^{i_z} z^{k-(j+1)}$  (resp.  $x^{i+i_y+1}, x^i y^{j-(k+1)}, x^i z^k$ ). Moreover, if  $j < k-1$  (resp.  $k < j-1$ ) it is primitive.*

*Proof.* Assume that  $j < k$ . The  $G$ -set  $T_{UR}^j(\Gamma)$  is spanned by  $x^i y^j, x^{i_z} z^{k-j}$  and it has one  $z$ -valley equal to  $x^{i_z}$  by Lemma (4.8). To finish the proof apply Lemma (4.7) to the  $G$ -set  $T_{UR}^j(\Gamma)$ .  $\square$

The preceding lemma allows us to define a sequence of primitive  $G$ -sets.

**Definition 5.6.** *If  $\Gamma_n$  is a primitive  $G$ -set we set:*

$$\Gamma_{n+1} = \begin{cases} T_U(T_{UR}^{j_n}(\Gamma_n)) & \text{if } j_n < k_n, \\ T_U(T_{UL}^{k_n}(\Gamma_n)) & \text{if } j_n > k_n, \end{cases}$$

where  $j_n, k_n$  denote the nonnegative numbers such that  $\Gamma_n$  is spanned by the monomials  $x^{i_n}, x^{i_{y,n}}y^{j_n}, x^{i_{z,n}}z^{k_n}$  for some  $i_n, i_{y,n}, i_{z,n} \geq 0$ .

Observe, that if  $j_n - k_n = \pm 1$  for some  $n$  then  $\Gamma_{n+1}$  is not primitive and the recursion stops.

**Corollary 5.7.** The numbers  $j_n, k_n$  satisfy the following formulas:

$$\begin{aligned} j_1 + 1 &= b, \\ k_1 + 1 &= r - b, \\ j_{n+1} + 1 &= \begin{cases} j_n + 1 & \text{if } j_n < k_n, \\ j_n + 1 - (k_n + 1) & \text{if } j_n > k_n, \end{cases} \\ k_{n+1} + 1 &= \begin{cases} k_n + 1 - (j_n + 1) & \text{if } j_n < k_n, \\ k_n + 1 & \text{if } j_n > k_n. \end{cases} \end{aligned}$$

Clearly, there is a direct link between the numbers  $j_n + 1, k_n + 1$  and the numbers appearing in the Euclidean algorithm for  $b$  and  $r - b$ . This relationship will be exploited later.

**Definition 5.8.** *Let  $\Theta(\Gamma)$  be a triangle of transformations of a  $G$ -set  $\Gamma$ . We define*

$$\tilde{\Theta}(\Gamma) = \bigcup_{\Gamma' \in \Theta(\Gamma)} \sigma(\Gamma')$$

to be the union of supports of the cones  $\sigma(\Gamma')$ , where  $\Gamma'$  runs through the  $G$ -sets in  $\Theta(\Gamma)$ .

To study the location of various cones in the fan  $\text{Hilb}^G \mathbb{C}^3$  it is convenient to give names to their rays.

**Definition 5.9.** Let  $\Gamma_n$  be the primitive  $G$ -set as defined in (5.6). Denote by  $\rho_n$  the common ray of the cones  $\tilde{\Theta}(\Gamma_n)$  and  $\sigma(\Gamma_n)$ .

Let  $\Gamma$  be any  $G$ -set. A ray of  $\sigma^\vee(\Gamma)$  will be called upper, (upper) left or right ray if it dual to the wall of  $\sigma(\Gamma)$  corresponding to the upper, (upper) left or right transformation, respectively.

**Remark 5.10.** Let  $\Gamma, \Gamma'$  be any two  $G$ -sets. Suppose that the cones  $\sigma(\Gamma)$  and  $\sigma(\Gamma')$  intersect either in a 2-dimensional face or in a ray. If the cones  $\sigma^\vee(\Gamma), \sigma^\vee(\Gamma')$  have a common ray  $\rho$  then there exists a 2-dimensional linear subspace of  $N \otimes \mathbb{R}$  containing a 2-dimensional face of  $\sigma(\Gamma)$  and of  $\sigma(\Gamma')$ , both of these dual to the ray  $\rho$ .

**Lemma 5.11.** For any  $G$ -set  $\Gamma$  with two valleys the set  $\tilde{\Theta}(\Gamma)$  is a rational simplicial cone.

*Proof.* Assume that  $G$ -set is spanned by the monomials  $x^i y^{j_y}, x^i z^{k_z}, x^{i_y} y^j, x^{i_z} z^k$  and let  $l = \min j, k, j - j_y, k - k_z$ . Because the upper right and upper left transformation commute (see Corollary (4.9)), by Remark (5.10) it is enough to establish the three following facts:

- the right rays of the cones  $\sigma^\vee(T_{UR}^n(\Gamma))$  for  $n = 0, \dots, l$  are the same,
- the left rays of the cones  $\sigma^\vee(T_{UL}^n(\Gamma))$  for  $n = 0, \dots, l$  are the same,
- the upper rays of the cones  $\sigma^\vee(T_{UR}^m(T_{UL}^n(\Gamma)))$  for  $m + n = l$  are the same.

These follow from Corollary (4.10).  $\square$

**Lemma 5.12.** Let  $\Gamma$  be a primitive  $G$ -set spanned by  $x^i, x^{i_y} y^j, x^{i_z} z^k$ . If  $j < k$  (resp.  $k < j$ ) then  $\mathbb{R}_+ e_2$  (resp.  $\mathbb{R}_+ e_3$ ) is a ray of  $\tilde{\Theta}(\Gamma)$ .

*Proof.* Suppose that  $j < k$ . The  $G$ -set  $\Gamma' = T_{UL}^j(\Gamma)$  is spanned by the monomials  $x^i z^{k_z+j}, x^{i_z} z^k$  and it has one valley (see Corollary (4.10)). The upper and left ray of  $\sigma(\Gamma')$  are equal to  $x^{i+1} z^{-k+k_z}$  and  $x^{-i+i_z} z^{k+1}$ , respectively. Evidently, the ray of  $\sigma^\vee(\Gamma')$ , dual to the 2-dimensional face of  $\sigma^\vee(\Gamma)$  spanned by the upper and left ray, is equal to  $\mathbb{R}_+ e_2$ .  $\square$

Note, that the cone  $\tilde{\Theta}(\Gamma_i)$  has, besides the ray common with  $\sigma(\Gamma_i)$ , two other rays: one equal to either  $e_2$  or  $e_3$  and the second which belongs to  $\sigma(\Gamma_{i+1})$ . We will investigate how the cones  $\tilde{\Theta}(\Gamma_i), \tilde{\Theta}(\Gamma_{i+1})$  fit together depending on the sign of  $(j_i - k_i)(j_{i+1} - k_{i+1})$ .

**Corollary 5.13.** Let  $\Gamma_n$  and  $\Gamma_{n+1}$  be two primitive  $G$ -sets. If  $(j_n - k_n)(j_{n+1} - k_{n+1}) > 0$  then the union of the supports of the cones  $\tilde{\Theta}(\Gamma_n), \tilde{\Theta}(\Gamma_{n+1})$  is a rational simplicial cone.

**Lemma 5.14.** *Let  $\Gamma_n$  and  $\Gamma_{n+1}$  be two primitive  $G$ -sets. Then  $\tilde{\Theta}(\Gamma_n) \cup \tilde{\Theta}(\Gamma_{n+1})$  is equal to the cone spanned by  $\rho_n, e_2, e_3$  minus (set-theoretical) the cone spanned by  $\rho_{n+1}, e_2, e_3$ .*

*Proof.* If  $(j_n - k_n)(j_{n+1} - k_{n+1}) > 0$  this follows from Corollary (5.13). Otherwise, the cones  $\tilde{\Theta}(\Gamma_n), \tilde{\Theta}(\Gamma_{n+1})$  have a common ray and a 2-dimensional face of  $\tilde{\Theta}(\Gamma_{n+1})$  is contained in a 2-dimensional face of  $\tilde{\Theta}(\Gamma_n)$ . To finish, note that  $e_2$  and  $e_3$  generate rays of  $\tilde{\Theta}(\Gamma_n)$  and  $\tilde{\Theta}(\Gamma_{n+1})$  (up to the order).  $\square$

Recall that  $\Gamma_l^{\text{yz}} = \text{span}(y^l, z^{r-l-1})$ . We will prove that the cones  $\sigma(\Gamma_l^{\text{yz}})$  fit nicely together with the cones  $\tilde{\Theta}(\Gamma_j)$  into the fan of  $\text{Hilb}^G \mathbb{C}^3$ .

**Lemma 5.15.** *The upper transformations of  $\Gamma_{b-1}^{\text{yz}}$  and  $\Gamma_b^{\text{yz}}$  coincide, where  $b \in \{1, \dots, r-1\}$  is an inverse of  $a$  modulo  $r$ . In fact, they are equal to  $\Gamma_1$ .*

*Proof.* By definition, the upper transformation of  $\Gamma_{b-1}^{\text{yz}}$  and  $\Gamma_b^{\text{yz}}$  replaces the monomial  $z^{r-b}$  and  $y^b$  with the monomial  $x$ , respectively.  $\square$

**Lemma 5.16.** *The upper rays of the cones  $\sigma^\vee(\Gamma_0^{\text{yz}}), \dots, \sigma^\vee(\Gamma_{b-1}^{\text{yz}})$  (resp.  $\sigma^\vee(\Gamma_b^{\text{yz}}), \dots, \sigma^\vee(\Gamma_{r-1}^{\text{yz}})$ ) are equal. The 1-dimensional cone  $\mathbb{R}_{\geq 0}e_1$  is a ray of each the cones  $\sigma(\Gamma_i^{\text{yz}})$ , for  $i = 0, \dots, r-1$ .*

*Proof.* The upper ray of the cones  $\sigma^\vee(\Gamma_0^{\text{yz}}), \dots, \sigma^\vee(\Gamma_{b-1}^{\text{yz}})$  is spanned by  $xz^{-r+b}$  and the upper ray of  $\sigma^\vee(\Gamma_b^{\text{yz}}), \dots, \sigma^\vee(\Gamma_{r-1}^{\text{yz}})$  is spanned by  $xy^b$ . The right and left rays of  $\sigma^\vee(\Gamma_l^{\text{yz}})$  are equal to  $y^{-l}z^{r-l}, y^{l+1}z^{r-l-1}$ , therefore  $\mathbb{R}_{+}e_1$  is a ray of  $\sigma(\Gamma_l^{\text{yz}})$ .  $\square$

**Corollary 5.17.** The sets

$$\bigcup_{l=0}^{b-1} \sigma(\Gamma_l^{\text{yz}}), \quad \bigcup_{l=b}^{r-1} \sigma(\Gamma_l^{\text{yz}})$$

are rational cones in  $N \otimes \mathbb{R}$  spanned by  $e_1, e_2, \rho_1$  and  $e_1, e_3, \rho_1$ , respectively.

*Proof.* This follows from Remark (5.10) and Lemma (5.16).  $\square$

## 6. MAIN THEOREM AND THE EUCLIDEAN ALGORITHM

By Theorem (3.10), when  $\Gamma$  varies through all  $G$ -sets, the cones  $\sigma(\Gamma)$  form a fan supported on the cone spanned by  $e_1, e_2, e_3$ . Therefore, it is enough to find  $G$ -sets different from the  $G$ -set  $\Gamma_l^{\text{yz}}$  which does not belong to any triangle of transformation. By looking at the supports of triangle transformations, it will turn out that those missing  $G$ -sets are exactly the upper transformations of the last  $G$ -set  $\Gamma_n$  defined in (5.6)). With the the help of the Euclidean algorithm we will be able to give a formula for a total number of  $G$ -set for fixed  $r$  and  $a$ .

**Definition 6.1.** Let  $m$  be an integer such that  $\Gamma_{m+1}$  is not primitive (i.e.  $\Gamma_m$  is the last primitive  $G$ -set in the sequence defined in (5.6)).

**Theorem 6.2** (Main Theorem). Let  $r, a$  be coprime natural numbers and let  $b$  be an inverse of  $a$  modulo  $r$ . Let  $G$  be a cyclic group of order  $r$ , acting on  $\mathbb{C}^3$  with weights  $1, a, r - a$ .

If  $\Gamma_1, \dots, \Gamma_{m+1}$  is the sequence from Definition (5.6) (that is  $\Gamma_n$  is a primitive  $G$ -set unless  $n = m + 1$ ) and if  $\Gamma_l^{\text{yz}} = \text{span}(y^{r-l-1}, z^l)$  then every  $G$ -set either

- belongs to a triangle of transformation of some  $\Gamma_n$  for  $n \leq m$ ,  
or
- is equal to a  $G$ -set  $\Gamma_l^{\text{yz}}$  for some  $l = 1, \dots, n$ , or
- is equal to an iterated upper transformation of the  $G$ -set  $\Gamma_{m+1}$ .

*Proof.* The proof uses Nakamura's Theorem (3.10), which asserts that the union of the supports of the cones  $\sigma(\Gamma)$  is equal to the positive octant in  $N \otimes \mathbb{R}$ . Lemma (5.14) and Corollary (5.17) combined imply that if a  $G$ -set  $\Gamma$  neither belongs to some triangle of transformation nor is equal to  $\Gamma_l^{\text{yz}}$  for some  $l$  then the cone  $\sigma(\Gamma)$  is supported in the cone spanned by  $e_2, e_3, \rho_{m+1}$ . On the other hand, the  $G$ -set  $\Gamma_{m+1}$  is equal either to  $\text{span}(x^{i_{m+1}}, x^{i_{y_{m+1}}} y^{j_{m+1}})$  or to  $\text{span}(x^{i_{m+1}}, x^{i_{z, m+1}} z^{k_{m+1}})$ , cf. Lemma (5.5). Therefore the  $j_{m+1}$ -th or  $k_{m+1}$ -th iterated upper transformation of  $\Gamma_{m+1}$  is equal to  $\Gamma^x = \text{span}(x^{r-1})$ . Moreover, the  $G$ -sets  $T_U^l(\Gamma_{m+1})$  and  $T_U^{l+1}(\Gamma_{m+1})$  satisfy assumptions of the Remark (5.10). This shows that the set

$$\bigcup_{l=0}^{\max\{j_{m+1}, k_{m+1}\}} \sigma(T_U^l(\Gamma_{m+1}))$$

is a cone generated by  $e_2, e_3, \rho_{m+1}$  which concludes the proof.  $\square$

**Remark.** The above theorem can be restated in a form of an algorithm computing the fan of the  $\text{Hilb}^G \mathbb{C}^3$  for fixed  $a$  and  $r$  (recall that the  $\text{Hilb}^G \mathbb{C}^3$  is normal, cf. Corollary (3.14)).

**Remark.** The two-stage construction of the  $\text{Hilb}^G \mathbb{C}^3$  for abelian subgroups in  $\text{SL}(3, \mathbb{C})$  by Craw and Reid in [CR02] appears to provide a coarse subdivision of the fan of the  $\text{Hilb}^G \mathbb{C}^3$  for the subgroup  $G$  in  $\text{GL}(3, \mathbb{C})$  of type  $\frac{1}{r}(1, a, r - a)$ . The coarse subdivision (i.e. with all interior lines of all triangles of transformations removed) is provided by the continued fraction expansions.

**Lemma 6.3.** Let  $p_i, q_i$  be the data of the Euclidean algorithm for the nonnegative integer numbers  $p_1, p_2$  with  $\text{GCD}(p_1, p_2) = p_{n+1}$ , that is

$$p_i = q_i p_{i+1} + p_{i+2}, \quad 0 < p_{i+2} < p_{i+1},$$

where  $p_{n+1} \neq 0$  and  $p_{n+2} = 0$ .

Then

$$\begin{aligned} \sum_{l=1}^n q_l p_{l+1} &= p_1 + p_2 - p_{n+1}, \\ \sum_{l=1}^n q_l p_{l+1}^2 &= p_1 p_2. \end{aligned}$$

**Theorem 6.4.** Fix some coprime numbers  $r$  and  $a$ . Let  $N$  denote the number of different  $G$ -sets for the action of type  $\frac{1}{r}(1, a, r - a)$ . Then

$$N = \frac{1}{2}(3r + b(r - b) - 1).$$

*Proof.* Denote  $\Gamma_l = \text{span}(x^{i_l}, x^{i_{y,l}} y^{j_l}, x^{i_{z,l}} z^{k_l})$ . The triangle of transformations of  $\Gamma_l$  consist of  $\binom{\min\{j_l+1, k_l+1\}+1}{2}$  cones (see Lemma (5.4)). Therefore

$$N = r + \max\{j_{m+1} + 1, k_{m+1} + 1\} + \sum_{l=1}^m \binom{\min\{j_l + 1, k_l + 1\} + 1}{2},$$

where the first two terms come from the  $G$ -sets  $\Gamma_l^{\text{yz}}$  and the consecutive upper transformations of  $\Gamma_{m+1}$ .

Suppose that  $b < r - b$ . Let the  $p_l$  and  $q_l$  be the data of the Euclidean algorithm for the coprime numbers  $p_1 = k_1 + 1 = r - b, p_2 = j_1 + 1 = b$  as in Lemma (6.3). Set  $q_0 = 1$ . In this notation, by the formulas from Corollary (5.7),

$$\begin{aligned} \min\{j_C + 1, k_C + 1\} &= p_D \\ \text{for } q_0 + \dots q_D &\leq C < q_0 + \dots q_{D+1}. \end{aligned}$$

Note that  $p_{n+1} = 1$  and  $q_n = \max\{j_{m+1} + 1, k_{m+1} + 1\}$ , thus  $N = r + q_n p_{n+1} + \frac{1}{2} \sum_{l=1}^{n-1} (q_l p_{l+1}^2 + q_l p_{l+1})$ . This, by simple computation, implies the assertion.  $\square$

## 7. EXAMPLE

By Theorem (6.2), for  $a = 5, r = 14$  every  $G$ -set, different from  $\Gamma_i^{\text{yz}}$ , belongs to a triangle of transformation of the primitive  $G$ -sets

$$\begin{aligned} \Gamma_1 &= \text{span}(x, y^2, z^{10}), \\ \Gamma_2 &= \text{span}(x^2, xy^2, z^7), \\ \Gamma_3 &= \text{span}(x^3, x^2 y^2, z^4), \\ \Gamma_4 &= \text{span}(x^4, x^3 y^2, z), \end{aligned}$$

or is an upper transformation of

$$T_U(\Gamma_5) = \Gamma^x.$$

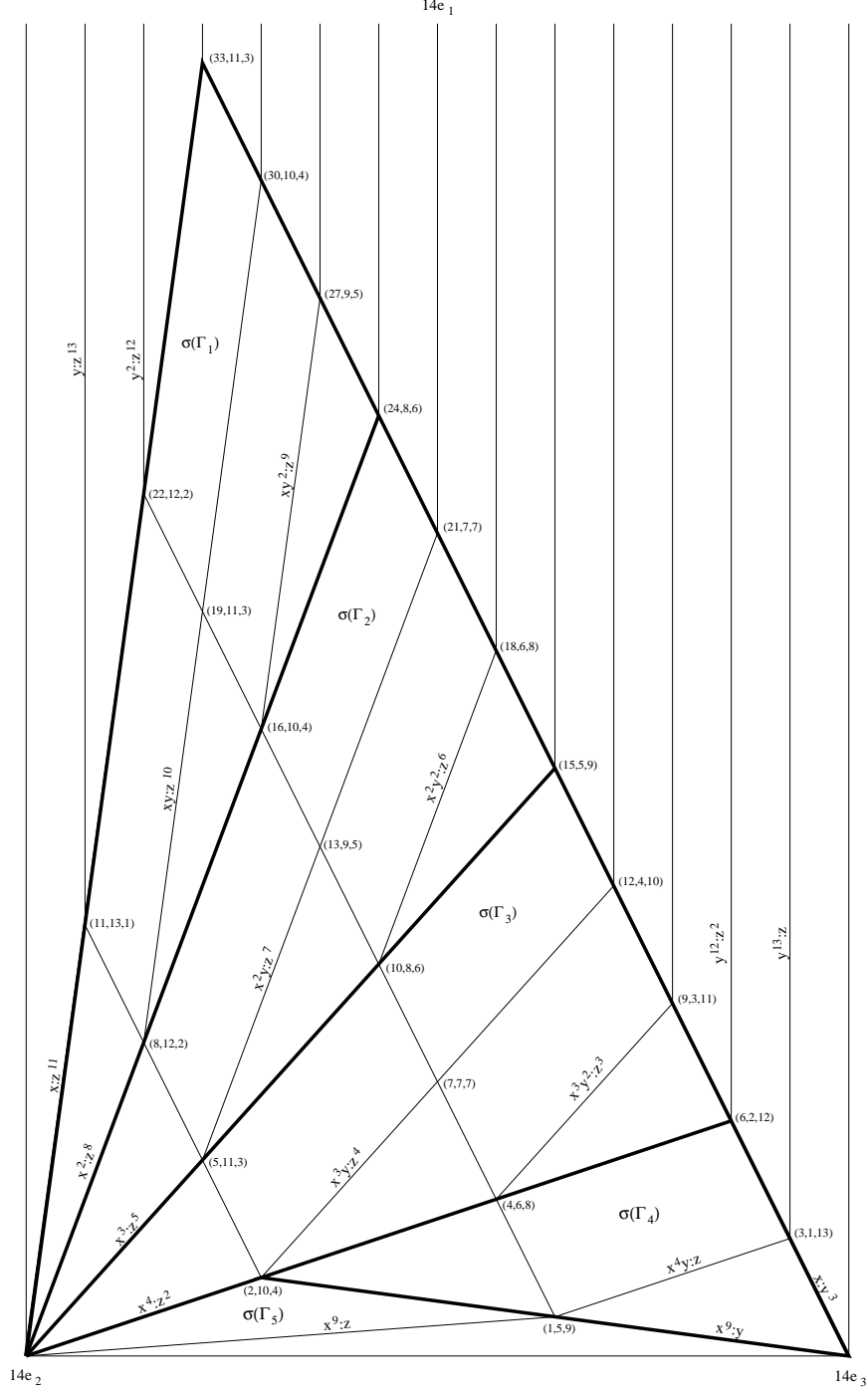


FIGURE 1. The fan of  $G$ -Hilb  $\mathbb{C}^3$  scheme for  $r = 14, a = 5$  intersected with hyperplane  $e_2^* + e_3^* = 14$ .

There are 37 different  $G$ -sets. Figure 1 shows the fan of  $G$ -Hilb  $\mathbb{C}^3$ , where  $e_1$  the ray generated by  $e_1$  is drawn at "infinity". The ratios along lines denote rays of the corresponding cones  $\sigma^\vee(\Gamma)$  (up to an



inverse in the multiplicative notation). Triangles of transformations are marked with thick line.

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